The Dubious Authority of Maxwell's Equations in Free Space

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Abstract. Because of the differential homogeneity of Maxwell's equations in free space, each solution can be perturbed in infinitely many ways into a physically unreasonable solution. Thus to be a solution holds no cachet. In particular, the tradtiional first example — the uniform plane wave — must be considered as only a metaphor. As an appendix, all uniform plane wave solutions are obtained.

A rite of passage. Generations of science and engineering students have labored through a course in electromagnetism, a course in three parts — electrostatics, where electric charges are held fixed; magnetics, where charges move at constant speed; and electromagnetics, where charges are accelerated.

Electrostatics has always been the most appealing to mathematicians, with its list of standard geometric problems of charge and conductor placements. The resulting electrostatic fields and their potentials are elegantly deduced through clever use of the divergence theorem.

In part two, magnetic fields are mapped by clever use of Stokes' theorem. These techniques and computations of the first two parts, will appear and reappear in subsequent (at first glance) unrelated courses in mechanics, heat transfer, partial differential equations, thermodynamics, chemistry, and so on. These classical ideas are the inspiration for much mathematics.

The triumphant third part of this traditional trilogy is electromagnetics, where by accelerating charges an experimenter can produce propagating wave fronts carrying energy that can do distant work. This electromagnetic phenomenon is believed to be completely described by the celebrated laws of Clerk Maxwell, first given as 20 equations in integral form, then soon after recast and condensed into four equations in differential form by Oliver Heaviside [**8**]. The publication of these four laws actually preceeded Hertz's demonstration of the physical existence of these propagating waves [**8**]. The four laws now hold a near mystical niche in the scientific pantheon, with extravagant praise such as "A glimpse into the mind of God" [**6**], or " Maxwell is the physicist's physicist" (Hawking), or "One scientific epoch ended and another began with James Clerk Maxwell" (Einstein). See [**5**]. Mawell's equations are thought to be an exact model, unlike (say) the heat equation, which although accurately models everyday heat transfer phenomenon, also predicts the unreasonable instantaneous transmission of information at an infinite distance [**4**], ex.1.35.

One commonally heard extravagant claim is central to this article: *"Whatever Maxwell allows, occurs,"* meaning that any mathematical solution, no matter how exotic — once the proper apparatus is constructed and experiment performed — will actually physically appear. But as we will now see, once an electromagnetic signal is emitted, it then propagates in (empty) free space where the four laws simplify and mathematical pathologies appear.

Maxwell's equations in free space. Because there are no free charges or current flow, Maxwell's equations reduce to four homogeneous linear equations: The pair (E, H) of vector fields, where E is the *electric* field and H is the *magnetic* field, satisfy

$$\nabla \cdot E = 0, \tag{1a}$$

$$\nabla \times H = \epsilon_0 \frac{\partial E}{\partial t}, \tag{1b}$$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}, \tag{1c}$$

$$\nabla \cdot H = 0. \tag{1d}$$

It is an entertaining exercise to deduce the following

Corollary. The vector fields E and H satisfy

$$\mu_0\epsilon_0 \frac{\partial^2 E}{\partial^2 t} = \nabla^2 E \quad \text{and} \quad \mu_0\epsilon_0 \frac{\partial^2 H}{\partial^2 t} = \nabla^2 H. \tag{2}$$

It is not our plan here to discuss the nature of the fundamental constants μ_0, ϵ_0, other than to note that they arose in many much earlier independent experiments of phenomena of electrostatics, magnetics, and optics. Thus (2) is the stunning prediction that the speed of propagation of all electromagnetic waves, from radio to light, travels at the identical speed of $c = 1/\sqrt{\mu_0\epsilon_0}$. See [**8**].

Traditionally, the first lecture on electromagnetic waves in free space is the derivation of uniform plane waves.

What is a uniform plane wave? A *plane wave* is a propagation mode, namely a solution pair (E, H) to Maxwell's four equations (1a)–(1d), for which there exists some non-zero constant vector P that is orthogonal to both E and H at each point (x, y, z) and moment of time t.This plane wave is *uniform* if for each t, E and H are both constant throughout the *propagating plane* attached at (x, y, z) perpendicular to P.

The unfortunate first free-space lecture. Invariably, plane waves in free space are derived by displaying Maxwell's equations simplified to their free space versions (1a)–(1d), then suddenly declaring that the electric field is some version of the complex exponential

$$E \;=\; E_0 \text{exp}[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{3}$$

[**1**] sec. 9.2; [**2**], sec. 7.1; [**7**] . It is then a quick few steps to deducing that such plane wave fields are uniform, sinusoidal, and that the electric and magnetic fields are perpendicular and form a right-handed trihedral with the direction of travel — *none of which is true for the general plane wave solution.* There is instead a rich zoo of plane wave solutions that are passed over by this traditional derivation. A careful derivation of all uniform plane waves is presented in the Appendix.

The canonical uniform plane wave. The first concrete example given of a plane wave is invariably the vector field pair

$$E(x,y,z,t) \ = \ E(z,t) \ = \ \sqrt{\mu_0}\sin(\omega t - kz)\mathbf{i} \qquad (4a)$$

and

$$H(x,y,z,t) \ = \ H(z,t) \ = \ \sqrt{\epsilon_0}\sin(\omega t - kz)\mathbf{j}, \qquad (4b)$$

where the *wave number* $k = \omega/c$

It is an entertaining project to verify that indeed this pair (E, H), propagating at speed $c = 1/\sqrt{\mu_0\epsilon_0}$ up the positive z-axis, staisfies the four free-space Maxwell equations (1a)–(1d). This (*linearly polarized*) solution is argueably the most common intuitive notion carried in the mind of every communications engineer. It, or a superposition with a quadrature version of itself, is the mode model of all transmitted radio, television, and optical communications [**3**]. Note that indeed E and H are orthogonal and form a right-handed trihedral with the direction of travel $\mathbf{k}$.

Embarrissingly, $(E, H + \mathbf{i})$ is also a solution pair *that is no longer perpendicular*. This trick has wider scope: Take any physically reasonable electromagnetic wave (E, H) that has now escaped into free space. At time t_0 suppose E points in the direction of the unit vector $\mathbf{u}$. Then the solution pair $(E, H + \mathbf{u})$ is not perpendiculat at this instant t_0.

Moreover, plane waves need not be uniform since we may perturb any solution pair by any time-invariant vector with divergence and curl 0, as for example by $y\mathbf{i} + x\mathbf{j}$.

More damaging yet, because of the small numerical values of μ_0 and ϵ_0 in the SI system of units, by perturbing both E and H of (4) by $\mathbf{i}$ *we obtain a trihedral with the direction of travel* $\mathbf{k}$ *that is at times left handed.* In summary,

Warning. Every solution pair (E, H) to Maxwell's equations in free space can be perturbed in infinitely many ways into a physically unreasonable solution pair. Thus being a solution to free-space Maxwell's equations carries no cachet.

The final humiliation of the uniform plane wave. There is a charming set of ideas surrounding the computation of the energy carried by an electromagnetic wave that enables it to affect a distant apparatus. These ideas are associated with John Poynting. Define the serendipitiously named *Poynting vector*

$$S \ = \ E \times H. \qquad (5)$$

Theorem. (Poynting)

$$\nabla \cdot S \ = \ -\frac{1}{2}\frac{\partial}{\partial t}\big(\epsilon_0 E^2 + \mu_0 H^2\big). \qquad (6)$$

Proof.

$$\nabla \cdot S = \nabla \cdot (E \times H) = (\nabla \times E) \cdot H - E \cdot (\nabla \times H)$$

$$= \ -\mu_0 \dot{H} \cdot H - E \cdot \epsilon_0 \dot{E} = -\frac{1}{2}\frac{\partial}{\partial t}\big(\mu_0 H^2 + \epsilon_0 E^2\big).$$

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The physical import of Poynting's theorem is as follows: Suppose Ω is a finite volume of space enclosed by its surface $\partial\Omega$. Then

$$V = \int_\Omega \frac{\mu_0 H^2 + \epsilon_0 E^2}{2} \, d\Omega \tag{7}$$

turns out to be the energy (Joules) stored within the volume Ω by the electric and magnetic fields at time t, [**2**], eqn. (6.106). Therefore via the divergence theorem, the flux

$$\int_{\partial\Omega} S \cdot \mathbf{n} \, d\sigma = \int_\Omega \nabla \cdot S \, d\Omega = -\frac{dV}{dt} \tag{8}$$

is the power (watts) exiting the volume Ω at time t.

Note that because the integrand of (7) — the energy density (Joules/m^3) — is non-negative, *the energy enclosed between any two propagating planes of a uniform plane wave is either zero or infinite*. Moreover, the net power leaving (or entering) the space between two such planes is infinite unless S is equal on both, whereupon there is no change in energy within. Thus the plane wave model does not reflect the finite energy that is found between expanding spherical wavefronts of actual electromagnetic emissions.

Summing up. Because any solution pair (E, H) to Maxwell's equations in free space can be additively perturbed in infinitely many ways to a solution pair with dubuous physical reality, the four equations alone are without authority. Some additional free-space conditions must be added to insure that solutions will be physically extant. In particular, the most familiar of all solutions, the linearly polarized sinusoidal travelling plane wave, must be considered at best a metaphor.

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Appendix: A characterization of uniform plane waves.

Theorem. Any twice-continuously differentiable uniform plane wave pair (E, H), after a rotation R of space about the origin, is of the form

$$RE \ = \ \sqrt{\mu_0} f(ct-z)\mathbf{i} \ + \ \sqrt{\mu_0} g(ct-z)\mathbf{j}, \tag{A1a}$$

$$RH \ = \ \big(a - \sqrt{\epsilon_0}\big) g(ct-z)\mathbf{i} \ + \ \big(b + \sqrt{\epsilon_0} f(ct-z)\big)\mathbf{j}. \tag{A1b}$$

Conversely, any continuously differentiable pair (E, H) of the form (A1) satisfies Maxwell's equations (1a) – (1d).

Proof. First rotate space about the origin so that the plane of propagation is perpendicular to the z-axis, i.e., $P = \mathbf{k}$. Thus we may assume $E = E(z,t)$ and $H = H(z,t)$. If need be, rotate again so that the propagation is in the direction of the *positive* z-axis. (The direction sense can be disambiguited by the time-averaged direction of the Poynting vector $S = E \times H$, [**3**], p. .)

Explicitly, let $E = (e_1, e_2, 0)$ and $H = (h_1, h_2, 0)$. Then we factor the pair (E, H) additively as follows:

$$E \ = \ E_0 \ + \ E_\perp \tag{A2a}$$

and

$$H \ = \ H_0 \ + \ H_\perp, \tag{A2b}$$

where

$$E_0 \ = \ (e_1, 0, 0), \qquad H_0 = (0, h_2, 0), \tag{A2c}$$

and

$$E_\perp \ = \ (0, e_2, 0), \qquad H_\perp = (h_1, 0, 0). \tag{A2d}$$

Then Maxwell's equations translate to the coordinate relations

$$\epsilon_0 \dot{e}_1 = -h_2', \ \epsilon_0 \dot{e}_2 = h_1', \ \mu_0 \dot{h}_1 = e_2', \ \mu_0 \dot{h}_2 = -e_1', \tag{A3}$$

where the overdot is partial differentiation with respect to time t, and where the prime is differentiation with respect to z. But then, the relations of (A3) are exactly Maxwell's equations (1a)–(1d) for the pairs (E_0, H_0) and $(E_\perp, H_\perp)$.

From (2), the wave equations must obtain, i.e.,

$$\ddot{e}_1 = c^2 e_1'' \tag{A4}$$

The general solution of (A4) is

$$e_1(z,t) \ = \ f(ct-z) \ + \ g(ct+z), \tag{A5}$$

the sum of two travelling waves, where f and g are any two twice continuously differentiable functions. (This famous result is obtained by an equally famous trick substitution $\alpha = ct - z$ and $\beta = ct + z$. See [**4**], ex.4.7.) But the plane of E_0 is travelling in the *positive* direction of the z-axis. Hence g is constant and may be absorbed into f. That is, after absorbing constants,

$$e_1 \;=\; \sqrt{\mu_0} f(ct - z). \tag{A5}$$

But applying Maxwell (1b) yields

$$h_2' \;=\; -\epsilon_0 \dot{e_1} \;=\; -\epsilon_0 c\sqrt{\mu_0} f'(ct-z) \;=\; -\sqrt{\epsilon_0} f'(ct-z). \tag{A6}$$

Integrating,

$$h_2 \;=\; \sqrt{\epsilon_0} f(ct-z) + b(t). \tag{A7}$$

But by Maxwell's (1c), $\mu_0 \dot{h}_2 \;=\; -e_1'$, i.e.,

$$\mu_0\sqrt{\epsilon_0} c f'(ct-z) \;+\; \dot{b}(t) \;=\; \sqrt{\mu_0} f'(ct-z) + \dot{b}(t)$$

$$= -e_1' = \sqrt{\mu_0} f'(ct-z), \tag{A8}$$

thus $b(t)$ is constant. Therefore

$$E_0 \;=\; \sqrt{\mu_0} f(ct-z)\mathbf{i} \quad \text{and} \quad H_0 \;=\; (\sqrt{\epsilon_0} f(ct-z) + b)\mathbf{j}. \tag{A9}$$

By a rotation of 90 degrees about the z-axis of the previous calculations, there is a twice-differentiable g and a constant a so that

$$H_\perp \;=\; (-\sqrt{\epsilon_0} g(ct-z) + a)\mathbf{i} \quad \text{and} \quad E_\perp \;=\; \sqrt{\mu_0} g(ct-z)\mathbf{j}. \tag{A10}$$

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Corollary. The electric and magnetic fields are orthogonal when and only when $af(ct-z) + bg(ct-z)$ is zero.

Corollary. E, H, and $P = \mathbf{k}$ form a right-handed trihedral whenever $bf(ct-z) - ag(ct-z)$ is nonnegative. There may be occasions when $E \times H$ points against the direction of propagation.